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NUMBER THEORY.

249. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A perfect number is a number which is equal to the sum of all its different divisors. In an old book on mathematics, the following method is given without proof for determining perfect numbers. The number $2^{n-1}(2^n - 1)$ is a perfect number if $2^n - 1$ is a prime number. Prove the formula.

250. Proposed by JOSEPH E. ROWE, State College, Pa.

Show by comparatively elementary means that the equation $x^{2n} + y^{2n} = z^{2n}$ is impossible of solution in positive integers x, y, z, and n, unless at least one of the integers $x, y, z \equiv 0 \pmod{3.}$. In particular, consider the case $n = 1$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

450. Proposed by J. E. ROWE, Pennsylvania State College.

If the four roots of the quartic equation, $A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, are so related that $B \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$, show by elementary algebra that two roots of A are real and two imaginary. Show also by means of elementary algebra that A cannot have two equal roots without having three, if the condition $B = 0$ is satisfied.

SOLUTION BY J. A. BULLARD, Worcester, Mass.

Let

$$A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 \equiv (ax^2 + 2bx + c)(a'x^2 + 2b'x + c') = 0.$$

Then

$$a_0 = a'a, \quad 2a_1 = a'b + ab', \quad 6a_2 = a'c + 4b'b + ac', \quad 2a_3 = b'c + bc' \quad \text{and} \quad a_4 = c'c;$$

whence,

$$\begin{aligned} a_0a_4 &= a'ac'c, \\ -4a_1a_3 &= -ab'^2c - a'b^2c' - a'b'bc - abb'c', \\ 3a_2^2 &= \tfrac{1}{12}(a'c)^2 + \tfrac{4}{3}(b'b)^2 + \tfrac{1}{12}(ac')^2 + \tfrac{2}{3}a'b'bc + \tfrac{2}{3}abb'c' + \tfrac{1}{6}aa'cc'. \end{aligned}$$

Adding these equations and simplifying, we have

$$\text{(I)} \qquad B \equiv (b^2 - ac)(b'^2 - a'c') + \tfrac{1}{3}[bb' - \tfrac{1}{2}(a'c + ac')]^2.$$

If the original coefficients are real numbers then a, b, c, a', b', c' can always be taken so as to have real values. If $B = 0$ and the roots are all distinct the product of the discriminants $b^2 - ac$ and $b'^2 - a'c'$ is negative and hence one discriminant is negative and the other positive. Thus two of the four roots of $A = 0$ are the real and unequal roots of one quadratic and the other two roots are the imaginary roots of the second quadratic.

If $B = 0$ and two roots are equal let us assume them to be the roots of $ax^2 + 2bx + c = 0$, that is, $b^2 - ac = 0$. Then from (I) it follows that $bb' - \frac{1}{2}(a'c + ac') = 0$. Solving the quadratics and substituting from the relations just stated, we find the roots to be $-\frac{b}{a}$, $-\frac{b}{a}$, $-\frac{b}{a}$, $-\frac{bc'}{a'c}$. Thus, if $B = 0$ and two roots are equal, three roots must be equal.

A further examination of (I) shows that

If $B < 0$, two roots are imaginary and two are real and unequal.

If $B = 0$, two roots are imaginary and two are real and unequal, or three are equal, or all are equal.

If $B > 0$, two are imaginary and two are equal, or all are imaginary, or all are real. In this case we may have two double roots.

Also solved by GEORGE W. HARTWELL and the PROPOSER.

451. Proposed by H. S. UHLER, Yale University.

Prove that

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$

SOLUTION BY W. E. MILNE, Bowdoin College.

Applying successively the formula $\sin 2\theta = 2\cos\theta\sin\theta$ we obtain

$$\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2}, \qquad = 4\cos\frac{x}{2}\cos\frac{x}{4}\sin\frac{x}{4}, \qquad = 8\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\sin\frac{x}{8}, \qquad \cdots,$$

and in general

$$\frac{\sin x}{2^n\sin(x/2^n)} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cos\frac{x}{2^n}. \tag{1}$$

Now $\lim\limits_{n=\infty} 2^n\sin(x/2^n) = x$. Since equation (1) holds for every positive integral value of n, and the left-hand side approaches a limit $\dfrac{\sin x}{x}$ as n becomes infinite, it follows that the right-hand side also approaches a limit, and the limits are equal. Therefore,

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$

Also solved by E. B. WILSON, PAUL CAPRON, HORACE OLSON, C. A. NICKLE, A. H. HOLMES, H. S. BROWN, E. H. VANCE, NORMAN ANNING, J. A. BULLARD, J. J. GINSBURG, C. K. ROBBINS, H. L. AGARD, H. R. HOWARD, O. S. ADAMS, I. A. BENNETT, J. A. CAPARO, and the PROPOSER.

GEOMETRY.

479. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the locus of the point whose polars (polar planes) with respect to two given conics (quadrics), are perpendicular to each other.

SOLUTION BY THE PROPOSER.

A. POLARS WITH RESPECT TO CONICS.

If P is a point of the required locus, its polars p_1, p_2 with respect to the two given conics (Σ_1), (Σ_2) meet the line at infinity in two points Q_1, Q_2 separated harmonically by the cyclical points $\mathfrak{J}$, $\mathfrak{J}'$ at infinity. The pole, with respect to (Σ_1), of any line passing through Q_1 lies on the polar q_1 of Q_1 with respect to (Σ_1); and the pole, with respect to (Σ_2), of any line passing through Q_2 lies on the polar q_2 of Q_2 with respect to (Σ_2). When Q_1 varies on the line at infinity, q_1 turns about the center C_1 of (Σ_1), the range (Q_1) and the pencil (q_1) being projective. Similarly for the range (Q_2) and the pencil (q_2). The points Q_1, Q_2 being a couple of conjugate elements in the involution whose double elements are the cyclical points $\mathfrak{J}$, $\mathfrak{J}'$, we have

$$(q_1\cdots)\barwedge(Q_1\cdots)\barwedge(Q_2\cdots)\barwedge(q_2\cdots)$$

Hence: *The locus of the point, whose polars, with respect to two given conics, are perpendicular to each other, is, in general, a conic passing through the centers of the given curves.*

I. Two central conics. If (Σ_1) and (Σ_2) are both central conics, let (C) be the circle having for diameter the line C_1C_2 of their centers. The couples of conjugate diameters of (Σ_1) determine an involution (I_1) of points on (C). Let O_1 denote the pole of this involution. The conic (Σ_2) gives rise to a similar involution (I_2) with O_2 as its pole. Let M be any point of (C). If M_1, M_2 are the two other points of intersection of (C) with the lines O_1M, O_2M, respectively, the point of intersection P of the diameters C_1M_1, C_2M_2 will belong to the required locus (Σ), because the polars of P with respect to (Σ_1) and (Σ_2) are parallel to the diameters C_1M and C_2M, respectively conjugate to C_1P, C_2P in the two given conics, and the two former are perpendicular to each other. This affords an expeditious way for the construction of (Σ).

The points of intersection N, N' of the circle (C) with the line O_1O_2 belong to the required locus (Σ), as is readily seen by making the point M coincide with each one of them in turn. These two points will necessarily be real, if at least one of the points O_1, O_2 lies inside of (C).

The double elements of (I_1) on (C) are the points of contact of the tangents to (C) drawn from O_1. The lines joining C_1 to these points of contact are the double elements of the involution of